Binary Relations

Part II

Outline for Today

- ***Proving an Equivalence Relation***
  - A proof that ~ is an equivalence relation
- ***Properties of Equivalence Relations***
  - What’s so special about those three rules?
- ***Cyclic Property***
  - How it relates to our other three properties, and equivalence relations

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

Equivalence Relation Proofs

- Let's suppose you've found a binary relation R over a set A and want to prove that it's an equivalence relation.
- How exactly would you go about doing this?

An Example Relation

- Consider the binary relation ~ defined over the set ℤ:

$$a \sim b \quad \text{if} \quad a + b \text{ is even}$$

- Some examples:

$$0 \sim 4 \qquad 1 \sim 9 \qquad 2 \sim 6 \qquad 5 \sim 5$$

- Turns out, this is an equivalence relation! Let's see how to prove it.

We can binary relations by giving a rule, like this:

$$a \sim b \quad \text{if} \quad \textit{some property of a and b holds}$$

This is the general template for defining a relation. Although we're using "if" rather than "iff" here, the two above statements are definitionally equivalent. For a variety of reasons, definitions are often introduced with "if" rather than "iff." Check the "Mathematical Vocabulary" handout for details.

What properties must ~ have to be an equivalence relation?

Reflexivity
Symmetry
Transitivity

Let's prove each property independently.

$a \sim b$ if $a+b$ is even

Lemma 1: The binary relation $\sim$ is reflexive.

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What is the formal definition of reflexivity?

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Therefore, we'll choose an arbitrary integer a, then go prove that $a \sim a$.

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ASSUME: Choose an arbitrary integer a.

WANT TO SHOW: We want to show that $a \sim a$.

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Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$.

***a*~*b* if *a*+*b* is even**

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Lemma 1: The binary relation $\sim$ is reflexive.

Proof: Consider an arbitrary $a \in \mathbb{Z}$. We need to prove that $a \sim a$. From the definition of the $\sim$ relation, this means that we need to prove that $a+a$ is even.

To see this, notice that $a+a = 2a$, so the sum $a+a$ can be written as $2k$ for some integer k (namely, a), so $a+a$ is even.

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Lemma 2: The binary relation ~ is symmetric.

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Lemma 2: The binary relation ~ is symmetric.

Which of the following works best as the opening ("assume" part) of this proof?

A. Consider any integers a and b. We will prove $a \sim b$ and $b \sim a$.
B. Pick $\forall a \in \mathbb{Z}$ and $\forall b \in \mathbb{Z}$. We will prove $a \sim b \rightarrow b \sim a$.
C. Consider any integers a and b where $a \sim b$ and $b \sim a$.
D. Consider any integer a where $a \sim a$.
E. The relation ~ is symmetric if for any $a, b \in \mathbb{Z}$, we have $a \sim b \rightarrow b \sim a$.
F. Consider any integers a and b where $a \sim b$. We will prove $b \sim a$.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

a~b **if** ***a+b* is even**

Lemma 2: The binary relation ~ is symmetric.

Proof:

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What is the formal definition of symmetry?

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$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ (a \sim b \to b \sim a)$$

Therefore, we'll choose arbitrary integers a and b where $a \sim b$, then prove that $b \sim a$.

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E. The relation $\sim$ is symmetric if for any $a, b \in \mathbb{Z}$, we have $a \sim b \rightarrow b \sim a$.
F. Consider any integers a and b where $a \sim b$. We will prove $b \sim a$.

$a \sim b$ if $a+b$ is even

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Since $a \sim b$, we know that $a+b$ is even.

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***$a \sim b$* if *$a+b$* is even**

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$$\forall a \in \mathbb{Z}.\ \forall b \in \mathbb{Z}.\ \forall c \in \mathbb{Z}.\ (a \sim b \land b \sim c \to a \sim c)$$

Therefore, we'll choose arbitrary integers a, b, and c where $a \sim b$ and $b \sim c$, then prove that $a \sim c$.

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So there is an integer r, namely $k+m-b$, such that $a+c = 2r$. Thus $a+c$ is even, so $a \sim c$, as required.

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An Observation

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The formal definition of reflexivity is given in first-order logic, but **this proof does not contain any first-order logic symbols!**

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Lemma 2: The binary relation ~ is symmetric.

Proof: Consider any integers a and b where $a \sim b$. We need to show that $b \sim a$.

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The formal definition of symmetry is given in first-order logic, but **this proof does not contain any first-order logic symbols!**

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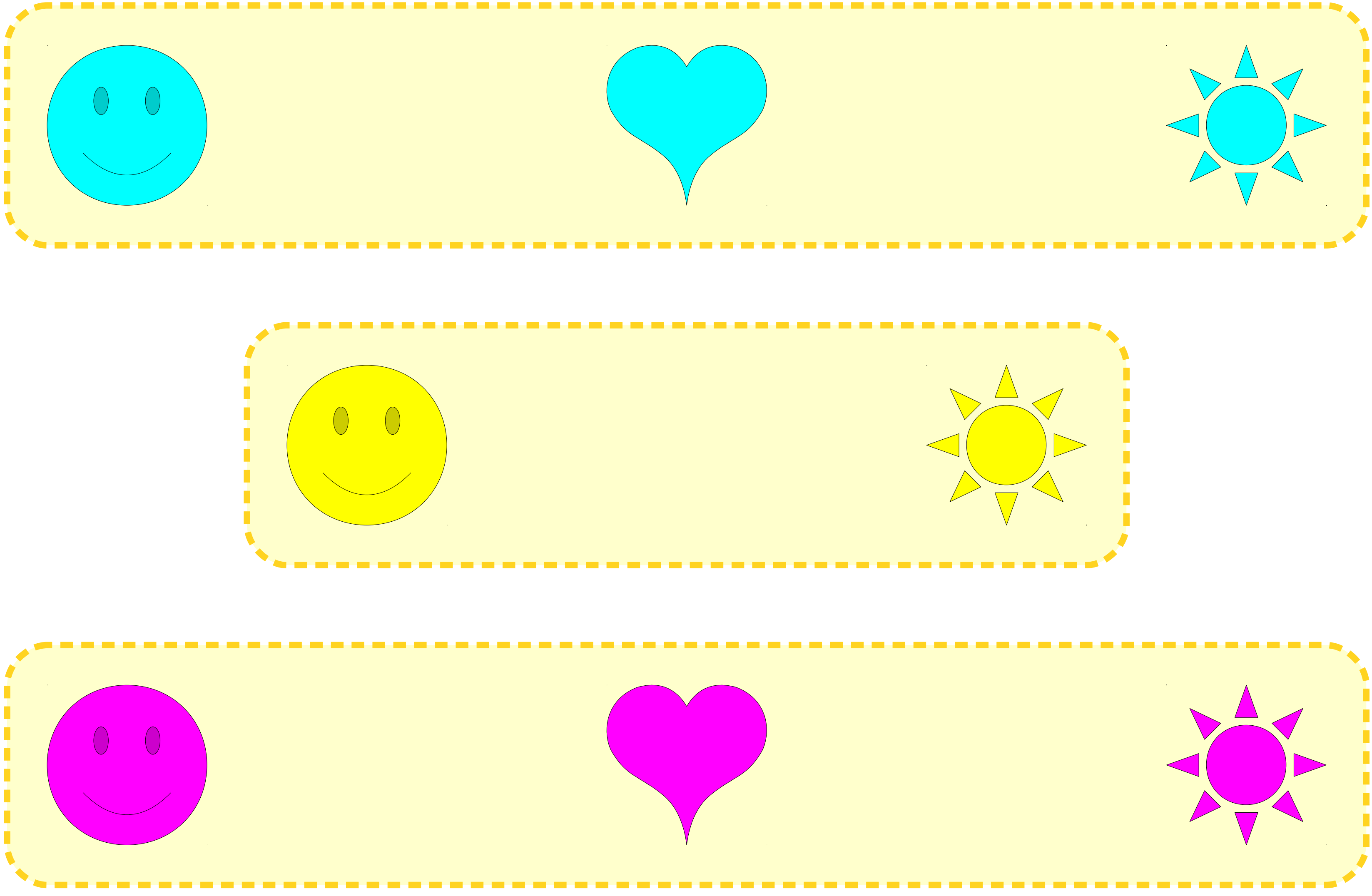
The formal definition of transitivity is given in first-order logic, but **this proof does not contain any first-order logic symbols!**

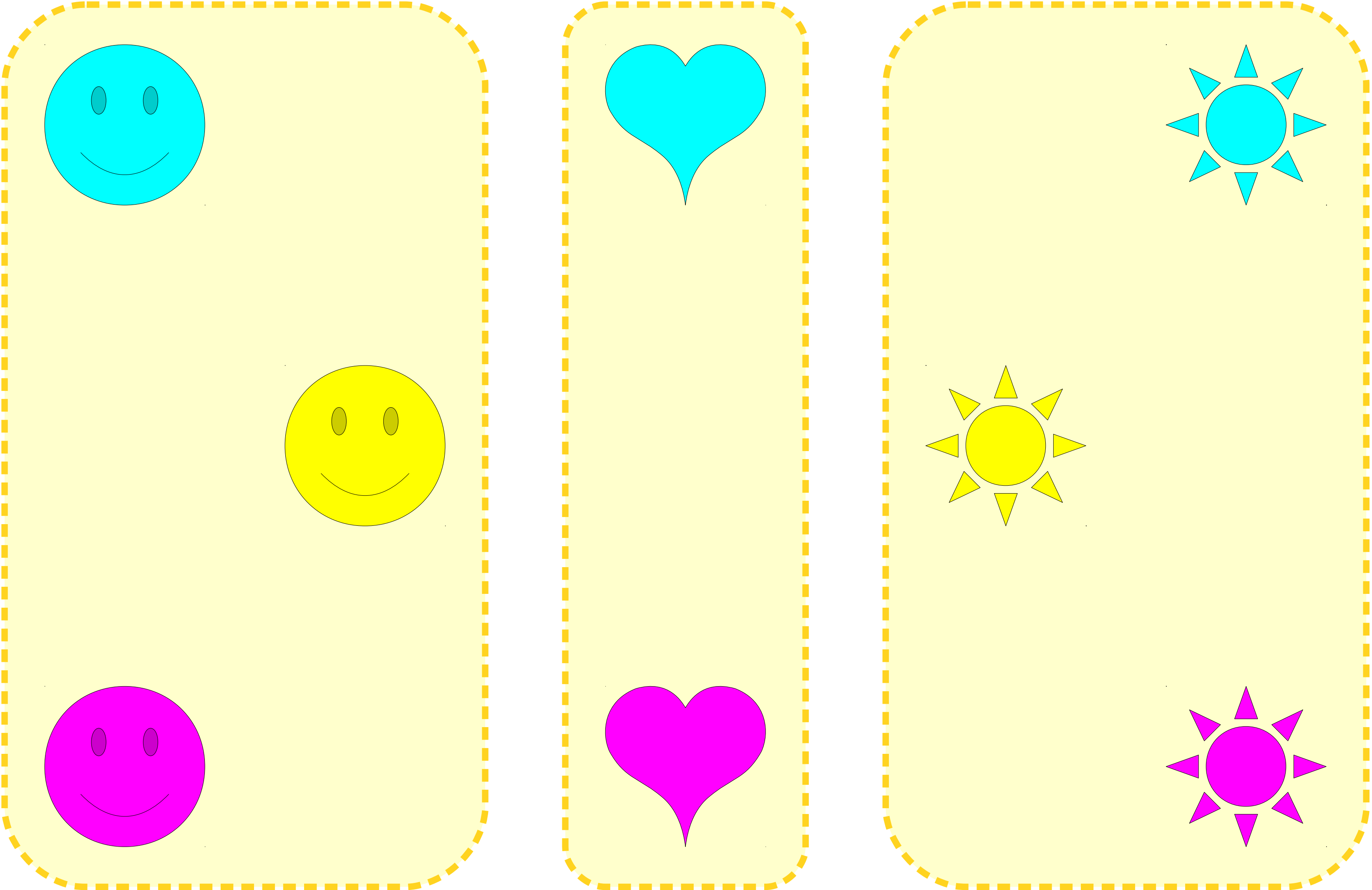
First-Order Logic and Proofs

- First-order logic is an excellent tool for giving formal definitions to key terms.
- While first-order logic *guides* the structure of proofs, it is *exceedingly rare* to see first-order logic in written proofs.
- Follow the example of these proofs:
  - Use the first-order logic definitions to identify your "**assume**" and "**want to show**" parts of the proof.
  - Write the proof in plain English using the conventions we set up in the first week of the class.

Properties of Equivalence Relations



xTy if x and y have the same color



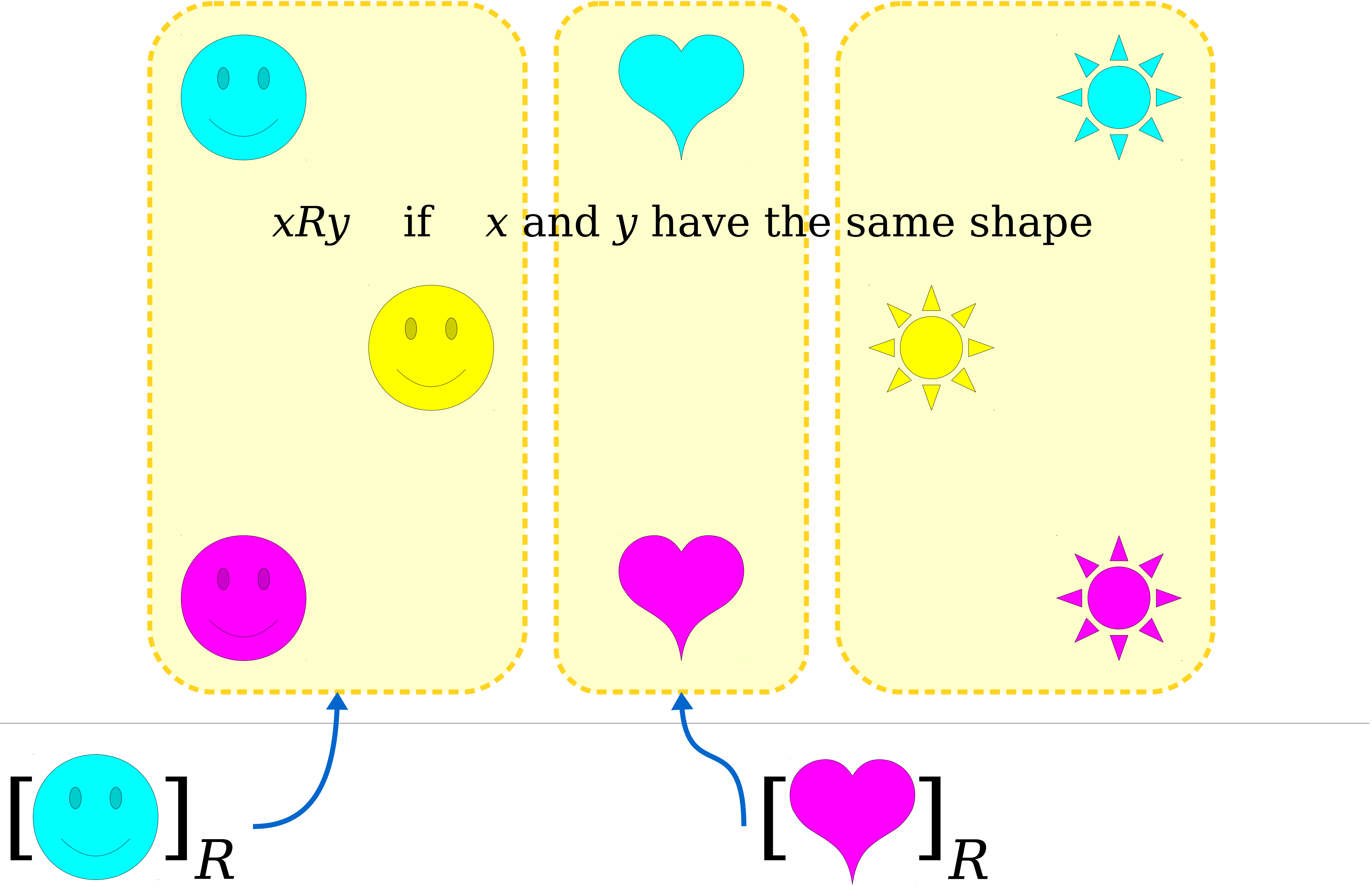
xRy if x and y have the same shape

Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

$$[x]_R = \{ \ y \in A \mid xRy \ \}$$

- Intuitively, the set $[x]_R$ contains all elements of A that are related to x by relation R.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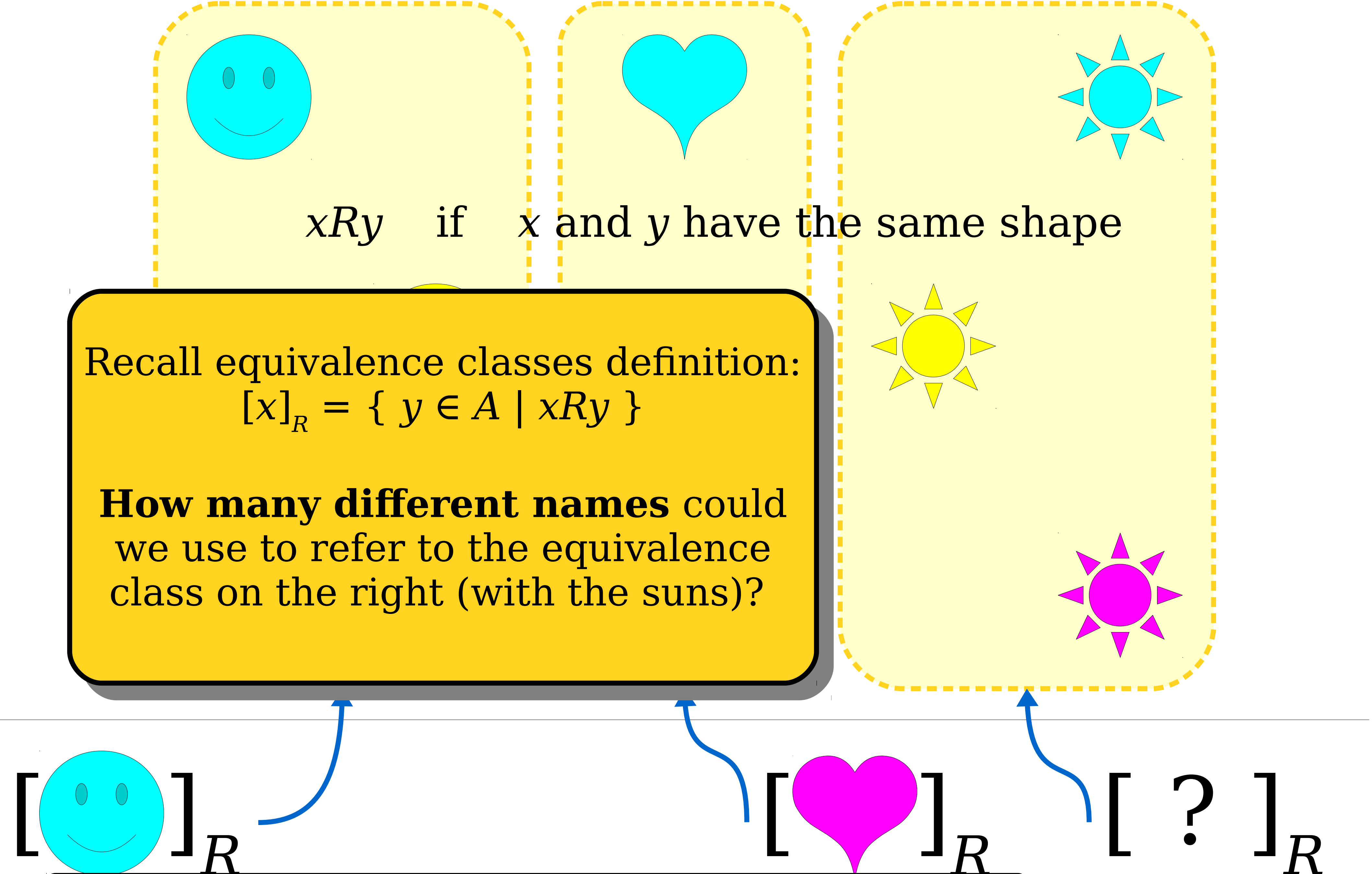

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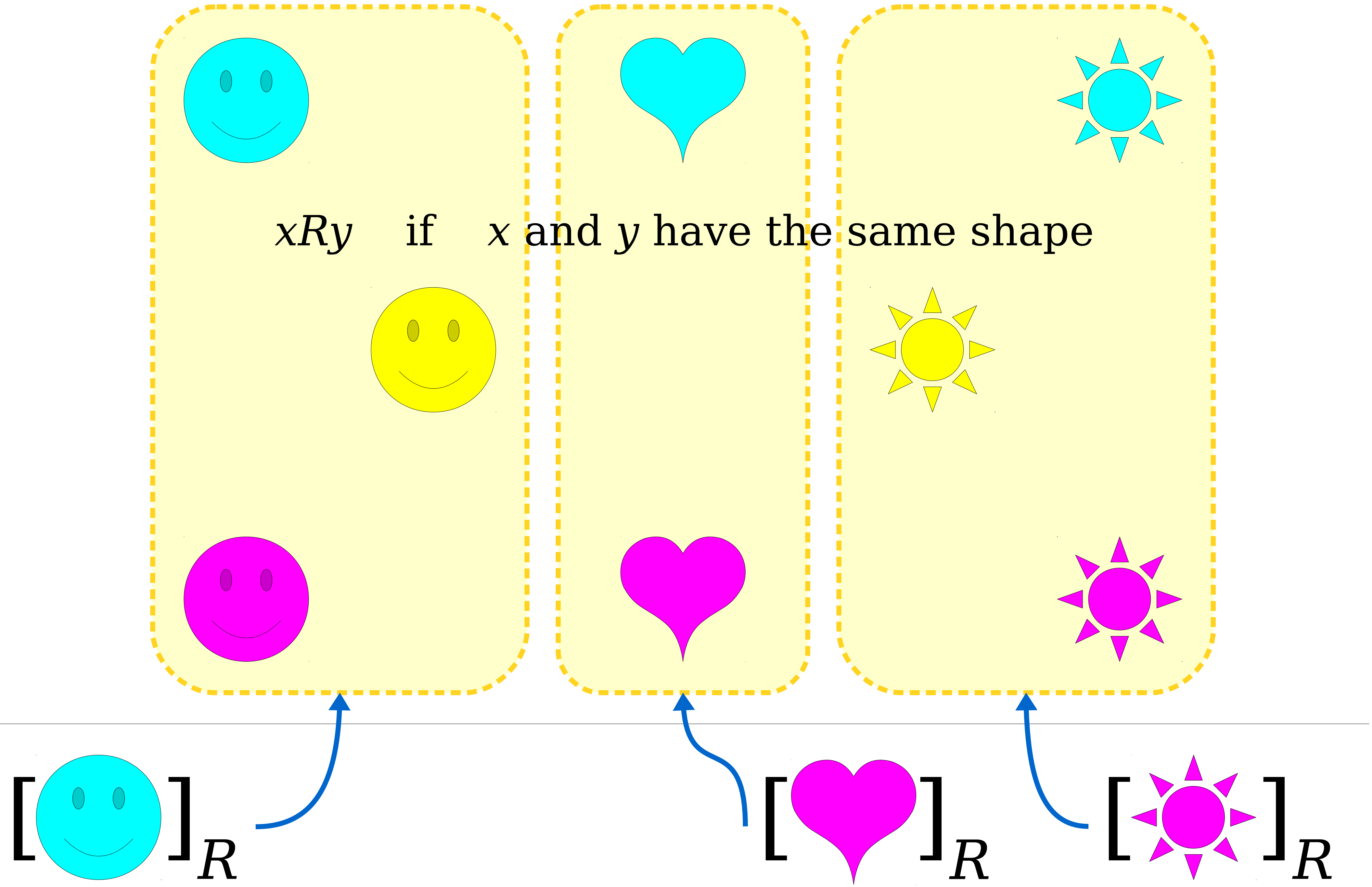
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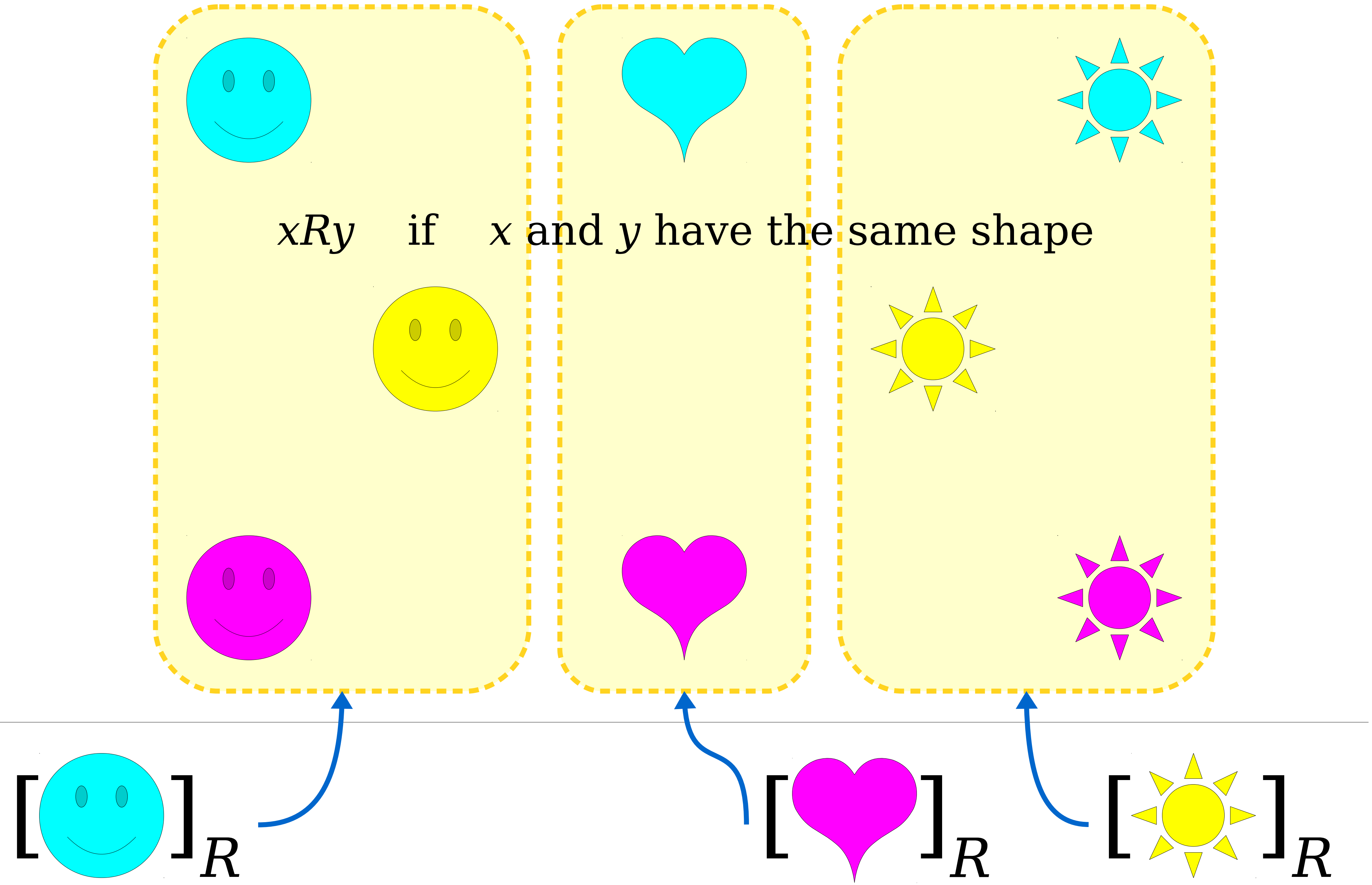

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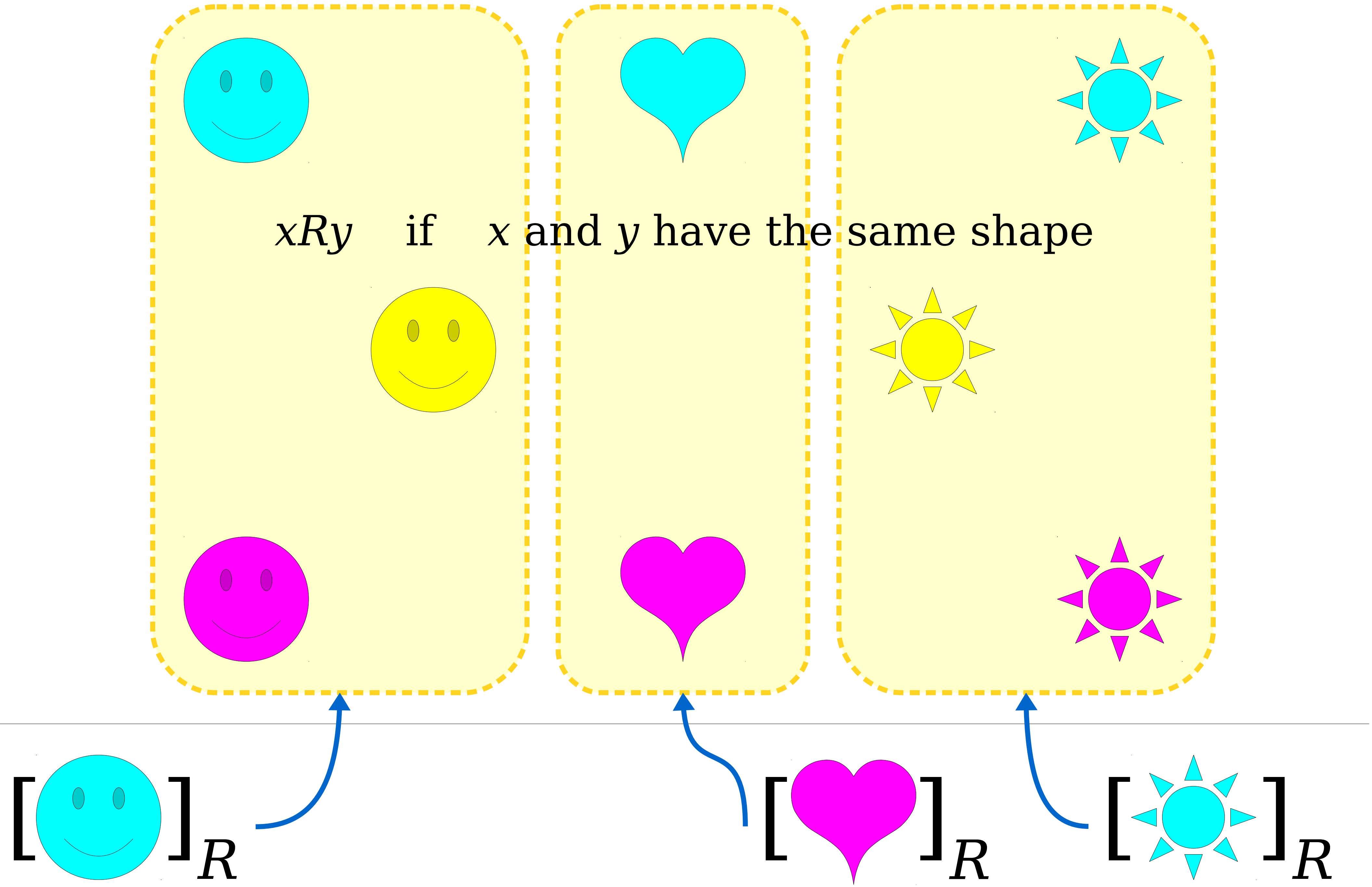
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The Fundamental Theorem of Equivalence Relations: Let R be an equivalence relation over a set A. Then every element $a \in A$ belongs to exactly one equivalence class of R.

How'd We Get Here?

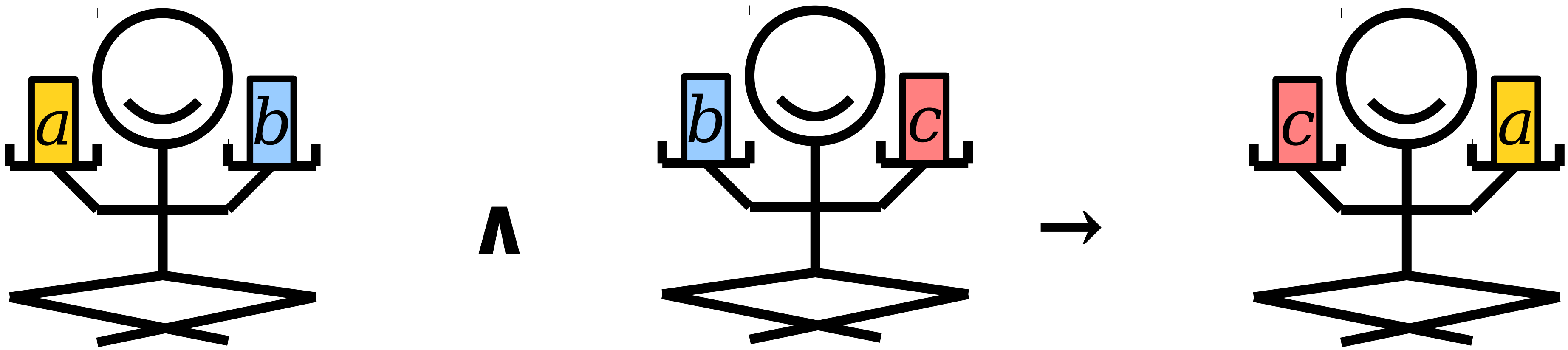
- We discovered equivalence relations by thinking about ***partitions*** of a set of elements.
- We saw that if we had a binary relation that tells us whether two elements are in the same group, it had to be reflexive, symmetric, and transitive.
- The FToER says that, in some sense, these rules precisely capture what it means to be a partition.
- ***Question:*** What's so special about these three rules?

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$$

A new rule that must be true:

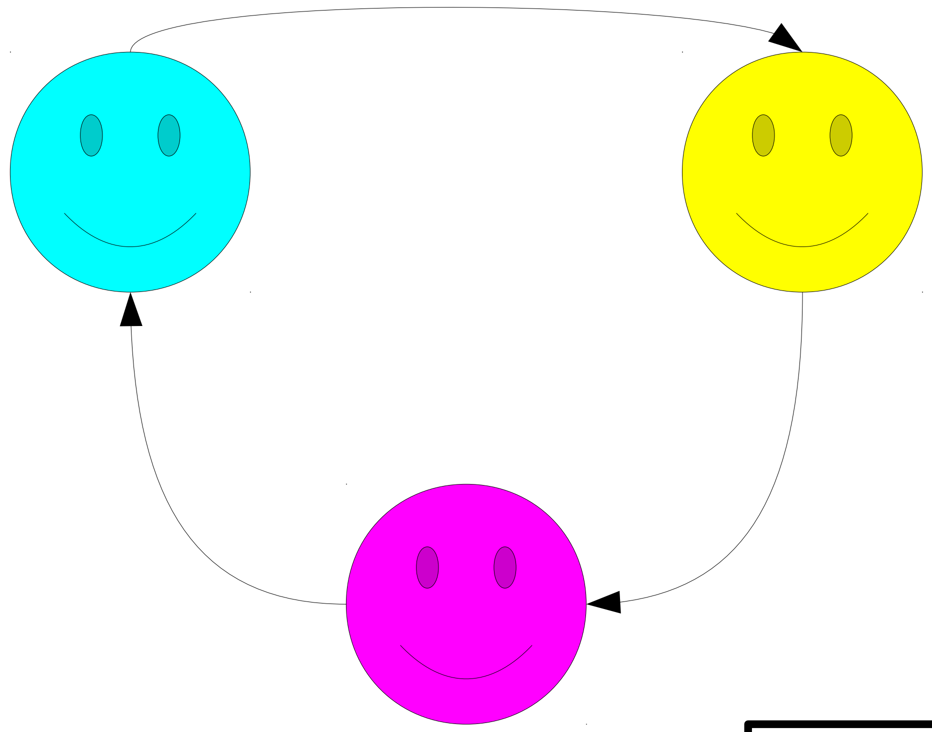


Relation this person holds: "Are these two things in the same partition?" for some mystery partition.

$$aRb \wedge bRc \rightarrow cRa$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to cRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow cRa)$$



A binary relation with this property is called ***cyclic***.

Theorem: A binary relation R over a set A is an equivalence relation if and only if it is reflexive and cyclic.

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Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

Lemma 2: If R is a binary relation over a set A that is reflexive and cyclic, then R is an equivalence relation.

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What We're Assuming

- R is an equivalence relation.
  - R is reflexive.
  - R is symmetric.
  - R is transitive.

What We Need To Show

- R is reflexive.
- R is cyclic.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

What We're Assuming	*What We Need To Show*
R is an equivalence relation. • R is reflexive. R is symmetric. R is transitive.	• R is reflexive. R is cyclic.

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	• R is reflexive.
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  - R is transitive.

What We Need To Show

R is reflexive.

R is cyclic.

- If aRb and bRc, then cRa.

Lemma 1: If R is an equivalence relation over a set A, then R is reflexive and cyclic.

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What We Need To Show

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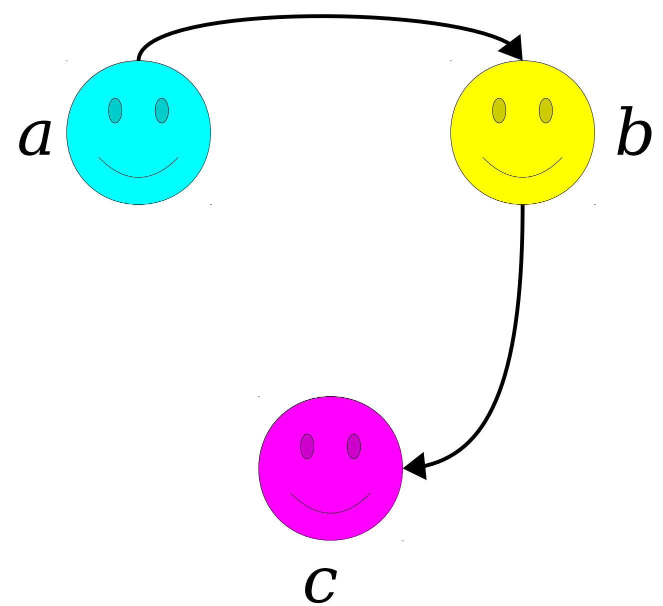

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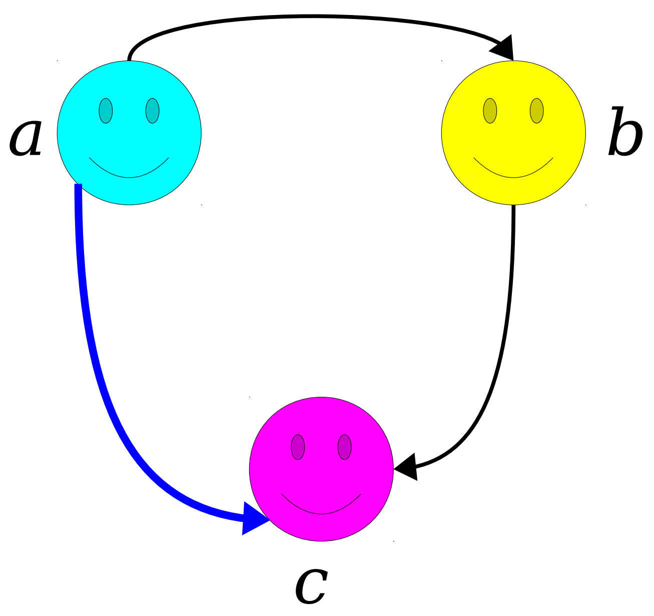

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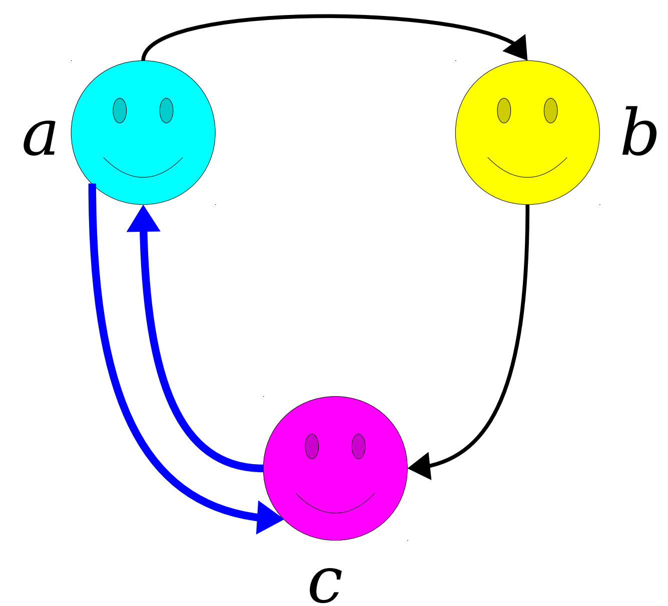

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You write the next sentence!
What is our assumption?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **your sentence**.

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Notice how the first few sentences of this proof mirror the structure of what needs to be proved. We're just following the templates from the first week of class!

To prove that R is … A
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Although this proof is deeply informed by the first-order definitions, notice that there is no first-order logic notation anywhere in the proof. That's normal – it's actually quite rare to see first-order logic in written proofs.

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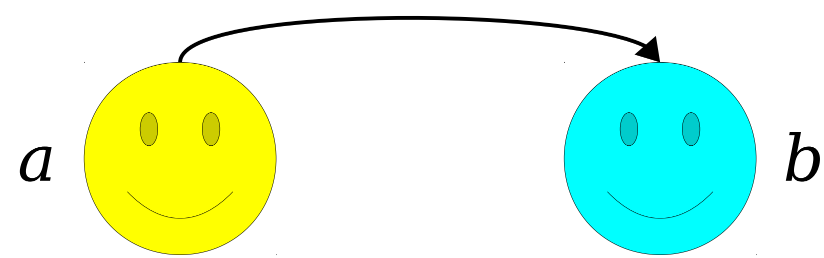

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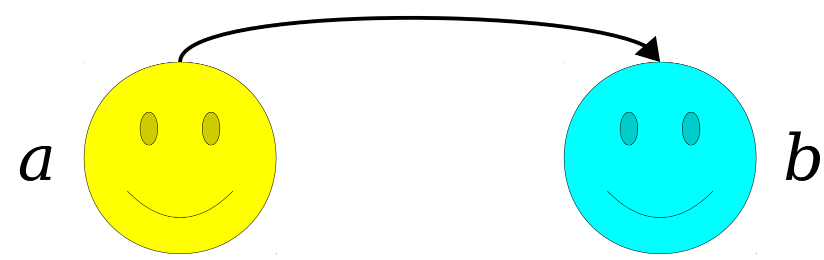

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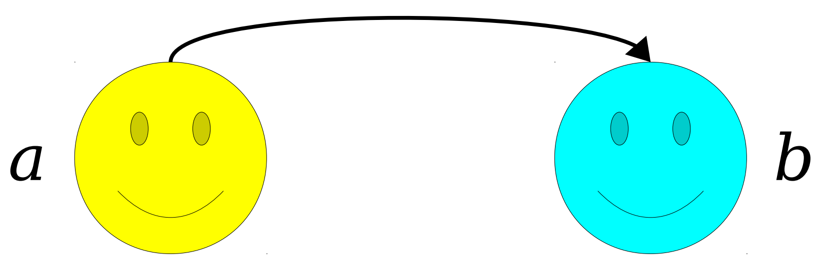

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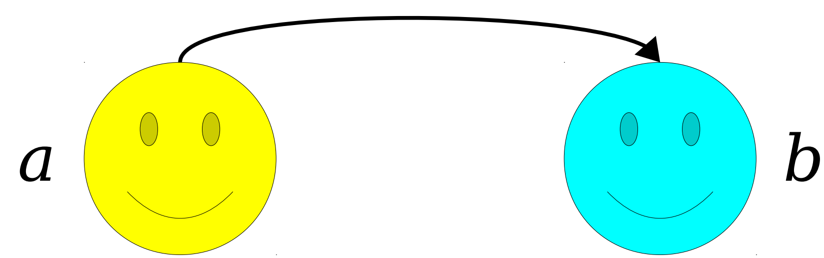

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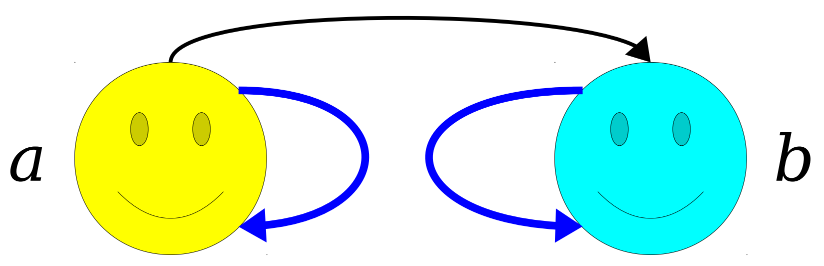

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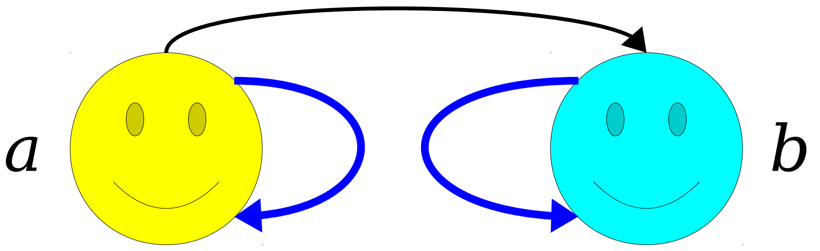

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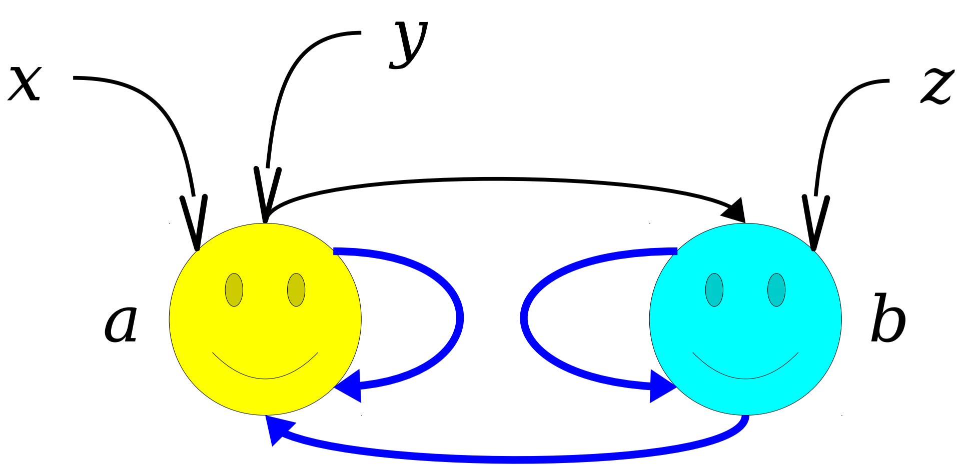

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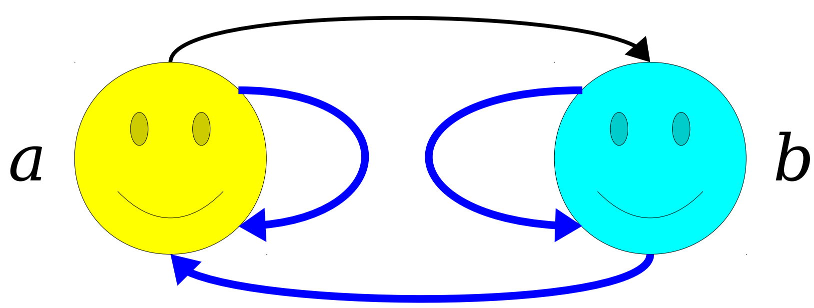

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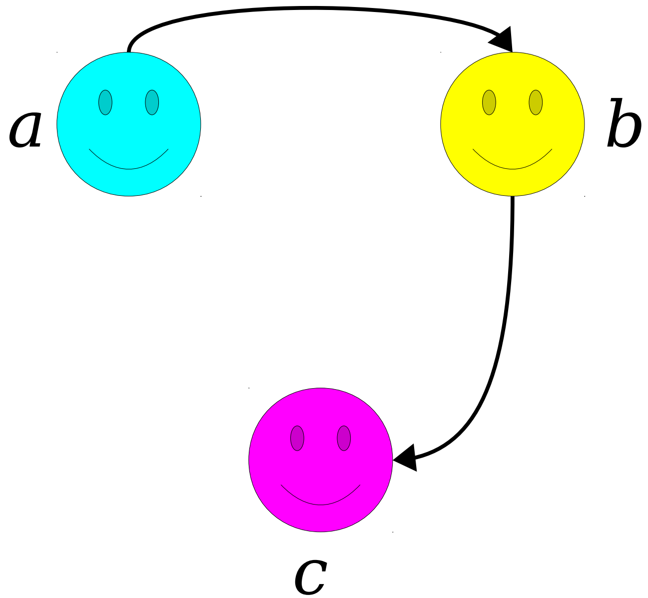

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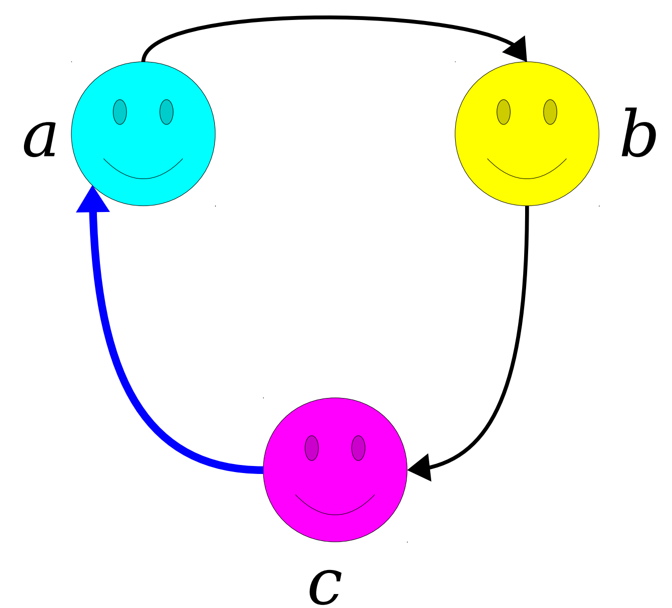

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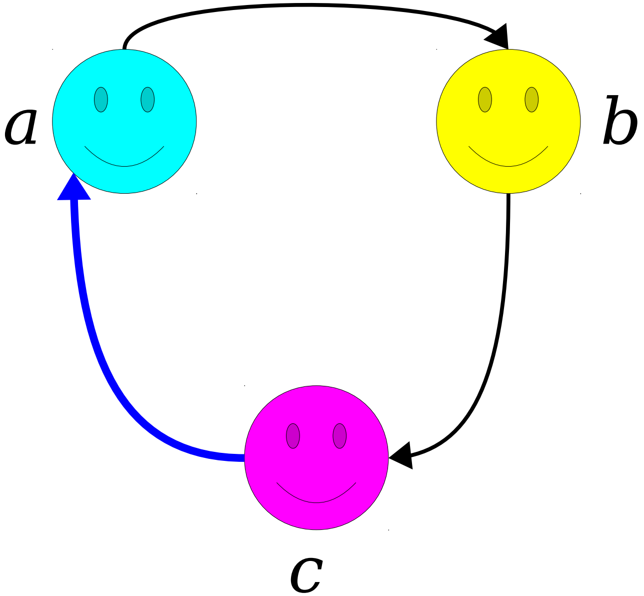

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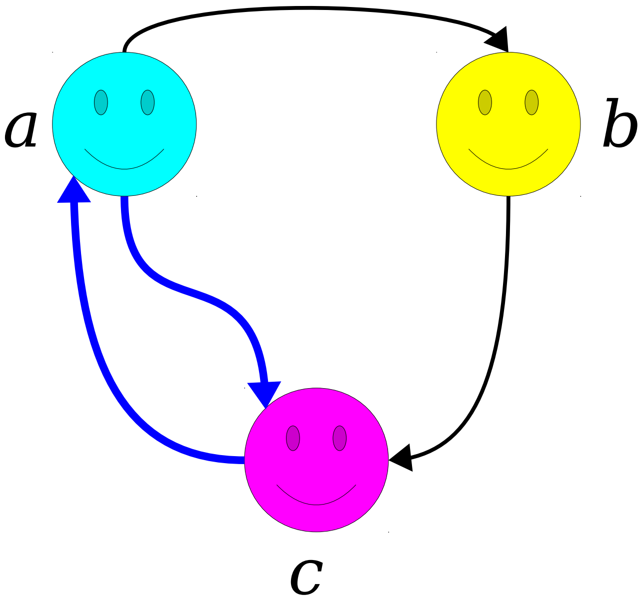

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Lemm … clic
anc …

Proof … that
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When writing proofs about terms with first-order definitions, it’s critical to call back to those definitions!

First, we'll prove that R is symmetric. To do so, pick any arbitrary $a, b \in A$ where aRb holds. We need to prove that bRa is true. Since R is reflexive, we know that aRa holds. Therefore, by cyclicity, since aRa and aRb, we learn that bRa, as required.

Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equi… is refle… know… show…

First … y arbit… at bRa … s. Then … t bRa, as required.

Notice how this setup mirrors the first-order definition of transitivity:

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

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Next, we'll prove that R is transitive. Let a, b, and c be any elements of A where aRb and bRc. We need to prove that aRc. Since R is cyclic, from aRb and bRc we see that cRa. Earlier, we showed that R is symmetric. Therefore, from cRa we see that aRc is true, as required. ■

Lemma 2: If R is a binary relation over a set A that is cyclic and reflexive, then R is an equivalence relation.

Proof: Let R be an arbitrary binary relation over a set A that is cyclic and reflexive. We need to prove that R is an equivalence relation. To do so, we need to show that R is reflexive, symmetric, and transitive. Since we already know by assumption that R is reflexive, we just need to show that R is symmetric and transitive.

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Next Time

- ***Functions***
  - How do we model transformations in a mathematical sense?
- ***Domains and Codomains***
  - Type theory meets mathematics!
- ***Injections, Surjections, and Bijections***
  - Three special classes of functions.